Finite quotient of abelian varieties with Calabi-Yau resolution

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Theorem [B84, GGK19, D18, HP19, BGL20]

Let X be a normal projective variety with klt singularities, with $c_1(K_X) = 0$. Then there is a finite quasiétale cover $q : \tilde{X} \to X$ such that

$$\tilde{X} = A \times \prod_i Y_i \times \prod_j Z_j,$$

where A is an abelian variety, Y_i is an irreducible holomorphic symplectic (IHS) variety with canonical singularities, Z_j is a Calabi-Yau (CY) variety with canonical singularities.

Three issues:

- The definition of singular IHS and CY varieties used in this theorem is hard to check in many instances.
- We do not know if singular CY of odd dimension have finite fundamental group.
- The number, nature, dimensions of the factors in the decomposition are not preserved by crepant birational maps.

A standard example where taking a crepant resolution alters the type of the Beauville-Bogomolov decomposition is given by a Kummer surface.

$$\begin{tikzcd}
\mathrm{Bl}_{A[2]}A \arrow[r, "q"] \arrow[d, "\eta"'] & S \arrow[d, "\varepsilon"] \\
A \arrow[r, "p"'] & A/\langle \pm \mathrm{id}_2 \rangle
\end{tikzcd}$$

where A is an abelian surface, $A[2]$ denotes its sixteen 2-torsion points, and S is the minimal resolution of singularities of $A/\langle \pm \mathrm{id}_2 \rangle$.

By construction, $\rho(S) \geq 17$, so S is a K3 surface.

This construction generalizes to produce a generalized Kummer variety as crepant resolution of A^{n-1} by the standard representation of the symmetric group S_n.

Pros: Computing Hodge numbers using representation theory of finite groups [AW'91]

Cons: Often no crepant resolution [R'87]

Produce interesting varieties with trivial canonical bundle using the following construction.

Let A be an abelian variety, and G be a finite group acting freely in codimension 1 on A. Depending on A and G, does A/G admit a crepant resolution?

- We can assume (and we will, throughout the talk) that G preserves the volume form on A.
- Up to changing A by an isogeny, we can assume that G contains no translations.
- If G acts freely, then the quotient is smooth; it is a hyperelliptic variety. Describing free actions on abelian varieties involves a lot of group theory, see [OS'01] or [D'22] for classification results in small dimension (17 possible groups in dimension 3, 79 in dimension 4).
- From now on, we focus on the case when G does not act freely.
- To come up with IHS manifolds in this way, we need every element in G to fix either nothing, or a union of abelian subvarieties of codimension 2 in A. That is hard to get.
- If we demand that G acts freely in codimension 2, there can be no IHS resolution.

Pros and cons of assuming freeness in codimension 2

Cons: Fewer examples under this assumption.

Pros:

- There are still examples satisfying it! (coming soon)
- A Calabi-Yau variety X is a crepant resolution of such a quotient iff it has a nef and big divisor D such that $D^{\dim X - 2} \cdot c_2(X) = 0$; that makes for a potentially interesting nef cone [W'91].
- If X is the unique crepant resolution of such a quotient A/G, then the automorphisms of A/G lift to automorphisms of X. Conversely, we hope due to [OS'01] that the map $X \to A/G$ is somewhat canonical, and allows to descend automorphisms of X to automorphisms of A/G. That makes for a potentially interesting automorphism group.

$j = e^{2i\pi/3}$, $u_7 = \frac{-1+i\sqrt{7}}{2}$.

E_τ is the elliptic curve with periods $1, \tau$.

M_7 is the following matrix. It has order 7 and eigenvalues $e^{2i\pi/7}, e^{4i\pi/7}, e^{8i\pi/7}$.

$$\begin{pmatrix} 0 & -8 & 7-10u_7 \\ 1 & -6-2u_7 & 11-u_7 \\ 0 & -1-2u_7 & 6+3u_7 \end{pmatrix}$$

Theorem [O'94]

Let A be an abelian threefold, G a finite group acting freely in codimension 2 on A. Suppose that A/G has a simply-connected crepant resolution.

Then A/G is isomorphic to either ${E_j}^3/\langle j\mathrm{id}_3\rangle$, or ${E_{u_7}}^3/\langle M_7\rangle$.

Some remarks:

- The two quotients each have a unique crepant resolution, that we denote X_3 and X_7 respectively. They are Calabi-Yau threefolds.
- Adding translations to G, we can derive some examples of crepant resolutions with non-trivial finite fundamental group too.
- Both X_3 and X_7 have non-trivial flops.
- Both X_3 and X_7 are rigid.
- Both X_3 and X_7 have infinite automorphism groups, so their nef cones cannot be rational polyhedral.

Theorem cod-3 [G'22]

Let A be an abelian variety, G a finite group acting freely in codimension 3 on A. Then A/G has a crepant resolution if and only if G acts freely on A.

Theorem dim-4 [G'22]

Let A be an abelian fourfold, G a finite group acting freely in codimension 2 on A. Then A/G has no simply-connected crepant resolution.

These theorems are consequences of a cascade of lemmas, that describe more and more restrictive necessary conditions on A and G for A/G to admit a simply-connected crepant resolution.

The main byproduct of this cascade of lemmas is the following result.

Theorem isog [G'22]

Let A be an abelian variety of dimension n, G a finite group acting freely in codimension 2 on A. Suppose that A/G admits a simply-connected crepant resolution that is a Calabi-Yau variety. Then one of the following two cases occurs.

	Case 1	Case 2
A is isogenous to	${E_j}^n$	${E_{u_7}}^n$
$\forall a \in A \; \exists 0 \leq k_a \leq \frac{n}{3}$ s.t. $\mathrm{Stab}(a)$ is isomorphic to	$(\mathbb{Z}/3\mathbb{Z})^{k_a}$	$(\mathbb{Z}/7\mathbb{Z})^{k_a}$
$\forall a \in A$ $\mathrm{Stab}(a)$ is generated by elements $g_i \in G$ of codiagonalizable matrices	$\mathrm{diag}(1_{3i-3}, j, j, j, 1_{n-3i})$	$\mathrm{diag}(1_{3i-3}, M_7, 1_{n-3i})$

Theorem isog is proved almost simultaneously with Theorem cod-3. The proof of Theorem dim-4 strongly relies on Theorem isog.

Theorem isog gives a clear understanding of the possible A, and of the local action of G on A.

Pros:

- The local actions are abelian. The quotient A/G has toroidal singularities.
- For A/G to admit a simply-connected resolution, G has to be generated by all the $\mathrm{Stab}(a)$ for a in A. Hence, G has a finite set of generators, which all have order 3, or all have order 7.

Cons: The group G may contain elements that do not fix any point. We have no control on them, as they do not intervene in any local action. In particular, G may have element of various orders other than 3 and 7, and it may be non-abelian.

A naive conjecture

Let A be an abelian variety, G be a finite group acting freely in codimension 2 on A. Suppose that A/G has a crepant resolution X. Then there is a finite étale cover $\tilde{X}$ of X such that

$$\tilde{X} = B \times {X_3}^k \times {X_7}^m,$$

where B is an abelian variety, X_3, X_7 are Oguiso's Calabi-Yau threefolds.

We use a mix of local and global arguments.

Typical local arguments involve the McKay correspondence.

Typical global arguments involve that G embeds in $\mathrm{Aut}(A)$, which, e.g., gives restrictions on the order of elements of G, and allows combinatorial arguments involving fixed loci of elements of G. In the case of Theorem isog, arguments involving the Calabi-Yau nature of the resolution (e.g., the scarcity of holomorphic forms) are of global sort.

Recall that G embeds in $\mathrm{Aut}(A)$, so every element g is of the form

$$g : [z] \in A \mapsto [M(g)z] + T(g) \in A,$$

with $M(g) \in \mathrm{GL}_n(\mathbb{C})$, the matrix of g, and $T(g) \in A$, the translation part of g.

Remark: M is a representation of G. Assuming G contains no translation, M is faithful. Of course, T is not a group homomorphism in general.

Fact [BL'92]: The characteristic polynomial of $M(g)\overline{M(g)} \in \mathrm{GL}_n(\mathbb{R})$ is unitary and has coefficients in $\mathbb{Q}$.

Corollary

If g is of finite order, then the characteristic polynomial of $M(g)\overline{M(g)}$ is a product of cyclotomic polynomials.

This is a global necessary condition on the faithful representation M of G.

The following result can be considered a starting point for McKay correspondence.

Theorem [IR'94]

Let H be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Consider a resolution Y of $\mathbb{C}^n/H$. The crepant exceptional divisors in Y are in one-to-one correspondence with the conjugacy classes of junior elements in H.

Definition

Let $M \in \mathrm{GL}_n(\mathbb{C})$ be a matrix of finite order d, with eigenvalues $e^{2i\pi a_1/d}, \ldots, e^{2i\pi a_n/d}$ for some parameters $0 \leq a_k \leq d-1$. We say that M is junior if

$$\mathrm{age}(M) := \frac{a_1 + \ldots + a_n}{d} = 1.$$

The exact nature of this one-to-one correspondence involves valuation theory. With the same tools, one can prove the following result.

Proposition

Let H be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Assume that $\mathbb{C}^n/H$ has a crepant resolution. Then H is generated by junior elements.

Note that $\mathbb{C}^n/H$ may have many non-junior elements.

We extend our definition of junior elements from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{Aut}(A)$.

Definition

Let A be an abelian variety, and $g \in \mathrm{Aut}(A)$. We say that g is junior if g fixes at least one point in A, and $M(g)$ is junior.

By the previous slide, we have an alternative: If G does not act freely on A, it contains at least one junior element.

Notation: $\omega = e^{i\pi/3}$.

Proposition

Let A be an abelian variety of dimension n, $g \in \mathrm{Aut}(A)$ a junior element such that $\langle g \rangle$ acts freely in codimension 2 on A. Then the vector of eigenvalues of $M(g)$ is one of the following:

- $(1_{n-3}, j, j, j)$
- $(1_{n-4}, i, i, i, i)$
- $(1_{n-4}, \omega, \omega, \omega, -1)$
- $(1_{n-5}, \omega, \omega, \omega, \omega, j)$
- $(1_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$
- $\left(1_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4\right)$
- $\left(1_{n-4}, \zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3\right)$
- $\left(1_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5\right)$
- $\left(1_{n-4}, \zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8\right)$
- $\left(1_{n-4}, \zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7\right)$
- $\left(1_{n-4}, \zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9\right)$
- $\left(1_{n-4}, \zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11}\right)$

Proof that there can be no junior element of prime order $p \geq 11$.

Take $g \in \mathrm{Aut}(A)$ junior of order p.

The characteristic polynomial Q of $M(g)\overline{M(g)}$ satisfies

$$Q = (X-1)^{\alpha}\Phi_p(X)^{\beta}.$$

Sharing the roots of Φ_p between $M(g)$ and $\overline{M(g)}$, we have

$$1 = \mathrm{age} M(g) \geq \frac{1+2+\ldots+\frac{p-1}{2}}{p} = \frac{p^2-1}{8p} \geq \frac{p-1}{8}.$$

Hence $p \leq 9$.

Now that we have some necessary conditions, we can hope to go the other way around and build new examples of Calabi-Yau varieties.

A naive idea at this point is to

- Take any junior matrix g in the previous list, other than the two matrices already used by Oguiso;
- Choose the dimension n so that g has no trivial eigenvalue;
- Use the theory of abelian varieties with CM-multiplication to find an abelian variety A of dimension n on which g can act;
- Note that $A/\langle g\rangle$ has reasonable enough Hodge numbers that its resolutions should not have intermediate degree differential forms;
- Finally try to find a crepant resolution of $A/\langle g\rangle$, e.g., by performing toric blow-ups of the singularities.

But the last step of this procedure will always fail! Let us explain why.

Technical lemma

Let A be an abelian variety, G a finite group acting freely in codimension 2 on A. Let $W \subset A$ be an abelian subvariety of codimension 3 or 4, such that

$$\mathrm{Pstab}(W) := \{g \in G \mid \forall w \in W,\ g(w) = w\}$$

is non-trivial. Then $\mathrm{Pstab}(W)$ is cyclic, and generated by one junior element.

Loosely speaking, this says that junior elements that are not powers of one another cannot interact much.

- Let B be a $\mathrm{Pstab}(W)$-equivariant complement of W in A. It has dimension 3 or 4. The group $\mathrm{Pstab}(W)$ embeds in the subgroup of $\mathrm{Aut}(B, 0_B)$ where 0_B is a point in $B \cap W$.
- Depending on the matrix of a junior element in $\mathrm{Pstab}(W)$, we can almost determine B with the theory of CM-multiplication.
- For example, if $\mathrm{Pstab}(W)$ contains a junior element of order 3, then there is a copy of ${E_j}^3$ inside B. Now, use freeness in codimension 2 to show that $\mathrm{Pstab}(W)$ is cyclic generated by one junior element.

- Back to the general case. Remark that $\mathrm{Pstab}(W)$ cannot have any element of prime order $p \geq 11$.
  Indeed, let $g \in \mathrm{Aut}(B, 0_B)$ be an element of order p.
  Then the characteristic polynomial of $M(g)\overline{M(g)}$ has degree $2\dim(B) \leq 8$, and is divisible by Φ_p which has degree $p-1$.
  So $p \leq 9$.
- Now we have $|\mathrm{Pstab}(W)| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta$ for some $\alpha, \beta, \gamma, \delta \geq 0$.
- Using our knowledge of B and Sylow theory, we show that $\alpha \leq 4, \beta \leq 1, \gamma \leq 1, \delta \leq 1$.
- Then we run a search through all the groups of order dividing 1680 in GAP, and check whether they can be generated by junior elements. The only solutions are the cyclic groups generated by one junior element.
- Caveat: We in fact do some more representation theory by hand before launching the computer search, to reduce possibilities, and thus computation time.

Now we can prove the following result.

Proposition

Let A be an abelian variety, G a finite group acting freely in codimension 2 on A. Suppose that A/G admits a crepant resolution. Then G contains no junior element of order 15.

- The same argument will work for orders 4, 8, 12, 16, 20, 24.
- The remaining three types of junior elements of order 6 are much harder to exclude. (But eventually, excluded all the same.) There are two reasons for that: Two of these three elements act freely in codimension 4, so they fall out of reach of the technical lemma. The third element behave badly with powers: Its square is a junior element, and its cube has strictly less distinct eigenvalues than itself.
- The junior elements of order 3 and 7 are of course not excluded.

Proof of the proposition. Let $g_{15} \in G$ be a junior element of order 15.

- It fixes an abelian subvariety $W \subset A$ of codimension 4.
- Let B be a $\mathrm{Pstab}(W)$-equivariant complement of W in A.
- g_{15} acts on B and has a composite order, so it fixes exactly one point in B, say 0_B.
- ${g_{15}}^3$ acts on B with order 5 and no trivial eigenvalue, so it fixes exactly 25 points. Among others it fixes $\tau \neq 0_B$.
- Let $W' = W + \tau - 0_B$. Then $\mathrm{Pstab}(W')$ thus contains a non-trivial element ${g_{15}}^3$. By the technical lemma it is generated be a junior element h.
- Let B' be a $\mathrm{Pstab}(W')$-equivariant complement of W' in A. By uniqueness in Poincaré's complete reducibility theorem, B and B' are isogenous.
- Using the theory of CM-multiplication again, we use that g_{15} has order 15 to entirely determine B. The isogeny type of B' is thus determined too. Running through the list of possible junior matrices for h, we see that the only junior that can act non-trivially on our pre-determined B' is that of order 15. So $h = h_{15}$.

- Now, since $\mathrm{Pstab}(W')$ is cyclic, thus abelian, $M({g_{15}}^3)$ and $M({h_{15}}^3)$ are codiagonalizable. Both g_{15}, h_{15}, and their cubes have exactly four distinct non-trivial eigenvalues, so $M(g_{15})$ and $M(h_{15})$ are codiagonalizable too.
- Let us show that g_{15} and h_{15} span the same group. Since the underlying group G contains no translation, M is a faithful representation and it is enough to prove that $M(g_{15})$ and $M(h_{15})$ span the same group.
- Write $M(g_{15}) = \mathrm{diag}(\zeta_{15}, {\zeta_{15}}^2, {\zeta_{15}}^4, {\zeta_{15}}^8)$. Up to replacing $M(h_{15})$ with its square, its fourth or eighth power, write $M(h_{15}) = \mathrm{diag}(\zeta_{15}, {\zeta_{15}}^a, {\zeta_{15}}^b, {\zeta_{15}}^c)$ with $\{a, b, c\} = \{2, 4, 8\}$.
- If $b \neq 4$, $M(g_{15})M(h_{15})^{-1}$ has both an eigenvalue of order 1 and an eigenvalue of order 15. Since Φ_{15} has degree 8, that contradicts the rationality requirement on the characteristic polynomial.
- If $b = 4$ and $a = 8$, then $M(g_{15})M(h_{15})^{-1} = \mathrm{diag}(1, {\zeta_5}^2, 1, {\zeta_5}^3)$. Again, that contradicts the rationality requirement on the characteristic polynomial.
- Hence, $M(g_{15})$ and $M(h_{15})$ span the same group. Hence, g_{15} is a power of h_{15}. In particular, g_{15} fixes $\tau \neq 0_B \in B$, final contradiction.

Let's go back and see how the pieces fit together.

Thanks for listening!